

# INTEGRATED AND DIFFERENTIATED SEQUENCE SPACES

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**ABSTRACT.** In this paper, we investigate integrated and differentiated sequence spaces which emerge from the concept of the space  $bv$  of sequences of bounded variation. The integrated and differentiated sequence spaces which was initiated by Goes and Goes [4]. The main propose of the present paper, we study of matrix domains and some properties of the integrated and differentiated sequence spaces. In section 3, we compute the alpha-, beta- and gamma duals of these spaces. Afterward, we characterize the matrix classes of these spaces with well-known sequence spaces.

## 1. INTRODUCTION

The set of all sequences denotes with  $\omega := \mathbb{C}^{\mathbb{N}} := \{x = (x_k) : x : \mathbb{N} \rightarrow \mathbb{C}, k \rightarrow x_k := x(k)\}$  where  $\mathbb{C}$  denotes the complex field and  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Each linear subspace of  $\omega$  (with the induced addition and scalar multiplication) is called a *sequence space*. The following subsets of  $\omega$  are obviously sequence spaces:  $\ell_\infty = \{x = (x_k) \in \omega : \sup_k |x_k| < \infty\}$ ,  $c = \{x = (x_k) \in \omega : \lim_k x_k \text{ exists}\}$ ,  $c_0 = \{x = (x_k) \in \omega : \lim_k x_k = 0\}$ ,  $bs = \{x = (x_k) \in \omega : \sup_n |\sum_{k=1}^n x_k| < \infty\}$ ,  $cs = \{x = (x_k) \in \omega : (\sum_{k=1}^n x_k) \in c\}$  and  $\ell_p = \{x = (x_k) \in \omega : \sum_k |x_k|^p < \infty, 1 \leq p < \infty\}$ . These sequence spaces are Banach space with the following norms;  $\|x\|_{\ell_\infty} = \sup_k |x_k|$ ,  $\|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^n x_k|$  and  $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$  as usual, respectively. And also the concept of integrated and differentiated sequence spaces was employed as  $\int X = \{x = (x_k) \in \omega : (kx_k) \in X\}$  and  $d(X) = \{x = (x_k) \in \omega : (k^{-1}x_k) \in X\}$  in [4].

A sequence, whose  $k$ -th term is  $x_k$ , is denoted by  $x$  or  $(x_k)$ . A *coordinate space* (or  $K$ -space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space  $X$  with a linear topology is called a  $K$ -space provided each of the maps  $p_i : X \rightarrow \mathbb{C}$  defined by  $p_i(x) = x_i$  is continuous for all  $i \in \mathbb{N}$ . A  $BK$ -space is a  $K$ -space, which is also a Banach space with continuous coordinate functionals  $f_k(x) = x_k$ , ( $k = 1, 2, \dots$ ). A  $K$ -space  $K$  is called an  $FK$ -space provided  $\lambda$  is a complete linear metric space. An  $FK$ -space whose topology is normable is called a  $BK$ -space. If a normed sequence space  $X$  contains a sequence  $(b_n)$  with the property that for every  $x \in X$  there is unique sequence of scalars  $(\alpha_n)$  such that

$$\lim_{n \rightarrow \infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \dots + \alpha_n b_n)\| = 0$$

then  $(b_n)$  is called *Schauder basis* (or briefly basis) for  $X$ . The series  $\sum \alpha_k b_k$  which has the sum  $x$  is then called the expansion of  $x$  with respect to  $(b_n)$ , and written as  $x = \sum \alpha_k b_k$ . An  $FK$ -space  $X$  is said to have  $AK$  property, if  $\phi \subset X$  and  $\{e^k\}$  is a basis for  $X$ , where  $e^k$  is a sequence whose only non-zero term is a 1 in  $k^{th}$  place for each  $k \in \mathbb{N}$  and  $\phi = \text{span}\{e^k\}$ , the set of all finitely non-zero sequences.

Let  $X$  and  $Y$  be two sequence spaces, and  $A = (a_{nk})$  be an infinite matrix of complex numbers  $a_{nk}$ , where  $k, n \in \mathbb{N}$ . Then, we say that  $A$  defines a matrix mapping from  $X$  into  $Y$ , and we denote it by writing  $A : X \rightarrow Y$  if for every sequence  $x = (x_k) \in X$ . The sequence  $Ax = \{(Ax)_n\}$ , the  $A$ -transform of  $x$ , is in  $Y$ ; where

$$(1.1) \quad (Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to  $\infty$ . By  $(X : Y)$ , we denote the class of all matrices  $A$  such that  $A : X \rightarrow Y$ . Thus,  $A \in (X : Y)$  if and only if the series on the right side of (1.1) converges for each  $n \in \mathbb{N}$  and each  $x \in X$  and we have  $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$  for all  $x \in X$ . A sequence  $x$  is said to be  $A$ -summable to  $l$  if  $Ax$  converges to  $l$  which is called the  $A$ -limit of  $x$ .

Let  $X$  is a sequence space and  $A$  is an infinite matrix. The sequence space

$$(1.2) \quad X_A = \{x = (x_k) \in \omega : Ax \in X\}$$

is called the matrix domain of  $X$  which is a sequence space (for several examples of matrix domains, see [2] p. 49-176). By  $\mathcal{F}$ , we will denote the collection of all finite subsets on  $\mathbb{N}$ . In [3], Başar and Altay have defined the

sequence space  $bv_p$  which consists of all sequences such that  $\Delta$ -transforms of them are in  $\ell_p$  where  $\Delta$  denotes the matrix  $\Delta = (\delta_{nk})$

$$\delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

for all  $k, n \in \mathbb{N}$ . And also we define the matrices  $\Gamma = (\gamma_{nk})$  and  $\Sigma = (\sigma_{nk})$  by

$$(1.3) \quad \gamma_{nk} = \begin{cases} k & , \quad (n = k) \\ -k & , \quad (n-1 = k) \\ 0 & , \quad (other) \end{cases}$$

$$(1.4) \quad \sigma_{nk} = \begin{cases} \frac{1}{k} & , \quad (n = k) \\ -\frac{1}{k} & , \quad (n-1 = k) \\ 0 & , \quad (other) \end{cases}$$

The integrated and differentiated sequence spaces which was initiated by Goes and Goes [4]. In the present paper, we study of matrix domains and some properties of the integrated and differentiated sequence spaces. In section 3, we compute the alpha-, beta- and gamma duals of these spaces. Afterward, we characterize matrix classes of these spaces with well-known sequence spaces.

## 2. THE SEQUENCE SPACES $\int bv$ AND $d(bv)$

The integrated spaces defined by

$$\begin{aligned} \int \ell_1 &= \left\{ x = (x_k) \in \omega : \sum_k |k.x_k| < \infty \right\} \\ \int bv &= \left\{ x = (x_k) \in \omega : \sum_k |k.x_k - (k-1).x_{k-1}| < \infty \right\} \end{aligned}$$

and the differentiated spaces defined by

$$\begin{aligned} d(\ell_1) &= \{x = (x_k) \in \omega : \sum_k |k^{-1}.x_k| < \infty\} \\ d(bv) &= \{x = (x_k) \in \omega : \sum_k |k^{-1}.x_k - (k-1)^{-1}.x_{k-1}| < \infty\}. \end{aligned}$$

Consider the notation (1.2) and the matrices (1.3), (1.4). From here, we can re-define the spaces  $\int bv$  and  $d(bv)$  by

$$(2.1) \quad \left( \int \ell_1 \right)_{\Delta} = \int bv \quad \text{or} \quad (\ell_1)_{\Gamma} = \int bv$$

and

$$(2.2) \quad [d(\ell_1)]_{\Delta} = d(bv) \quad \text{or} \quad (\ell_1)_{\Sigma} = d(bv).$$

Let  $x = (x_k) \in \int bv$  and  $\Delta x_k = x_k - x_{k-1}$ . The  $\Gamma$ -transform of a sequence  $x = (x_k)$  is defined by

$$(2.3) \quad y_k = (\Gamma x)_k = \begin{cases} x_1 & , \quad k = 1 \\ \Delta(kx_k) & , \quad k \geq 2 \end{cases}$$

where  $\Gamma$  is defined by (1.3). Let  $x = (x_k) \in d(bv)$  and  $\Delta x_k = x_k - x_{k-1}$ . The  $\Sigma$ -transform of a sequence  $x = (x_k)$  is defined by

$$(2.4) \quad y_k = (\Sigma x)_k = \begin{cases} x_2/2 & , \quad k = 2 \\ \Delta(k^{-1}x_k) & , \quad k \geq 3 \end{cases}$$

where  $\Sigma$  is defined by (1.4).

**Theorem 2.1.** *The spaces  $\int \ell_1$  and  $d(\ell_1)$  are BK-spaces with the norms  $\|x\|_{\int \ell_1} = \sum_k |kx_k|$  and  $\|x\|_{d(\ell_1)} = \sum_k |k^{-1}x_k|$ , respectively.*

*Proof.* Let  $x = (x_k) \in \int \ell_1$ . We define  $f_k(x) = x_k$  for all  $k \in \mathbb{N}$ . Then, we have

$$\|x\|_{\int \ell_1} = 1.|x_1| + 2.|x_2| + 3.|x_3| + \cdots + k.|x_k| + \cdots$$

Hence  $k.|x_k| \leq \|x\|_{\int \ell_1} \Rightarrow |x_k| \leq K.\|x\|_{\int \ell_1} \Rightarrow |f_k(x)| \leq K.\|x\|_{\int \ell_1}$ . Then,  $f_k$  is continuous linear functional for each  $k$ . Thus  $\int \ell_1$  is a BK-space.

In a similar way, we can prove that the space  $d(\ell_1)$  is a BK-spaces. □

**Lemma 2.2.** [4] *The space  $\int bv$  is a  $BK$ -space with the norm  $\|x\|_{\int bv} = \sum_k |\Delta(kx_k)|$ .*

**Theorem 2.3.** *The space  $d(bv)$  is a  $BK$ -space with the norm  $\|x\|_{d(bv)} = \sum_k |\Delta(k^{-1}x_k)|$ .*

*Proof.* Since  $d(bv) = [d(\ell_1)]_\Delta$  holds,  $d(\ell_1)$  is a  $BK$ -space with the norm  $\|x\|_{d(\ell_1)}$  and the matrix  $\Delta$  is a triangle matrix, then Theorem 4.3.2 of Wilansky[6] gives the fact that the space  $d(bv)$  is a  $BK$ -space.  $\square$

**Theorem 2.4.** (i). *The spaces  $\int \ell_1$  and  $d(\ell_1)$  have  $AK$ -property.*

(ii). *The spaces  $\int bv$  and  $d(bv)$  have  $AK$ -property.*

*Proof.* The fact that of the space  $\int bv$  has  $AK$ -property was given by Goes and Goes[4]. Then, we will only prove that the space  $d(bv)$  has  $AK$ -property in (ii).

Let  $x = (x_k) \in d(bv)$  and  $x^{[n]} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$ . Hence,

$$x - x^{[n]} = \{0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots\} \Rightarrow \|x - x^{[n]}\|_{d(bv)} = \|0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots\|$$

and since  $x \in d(bv)$ ,

$$\begin{aligned} \|x - x^{[n]}\|_{d(bv)} &= \sum_{k \geq n+1} |\Delta(k^{-1}x_k)| \rightarrow 0 \text{ as } n \rightarrow \infty \\ \Rightarrow \lim_{n \rightarrow \infty} \|x - x^{[n]}\|_{d(bv)} &= 0 \Rightarrow x^{[n]} \rightarrow x \text{ as } n \rightarrow \infty \text{ in } d(bv). \end{aligned}$$

Then the space  $d(bv)$  has  $AK$ -property.  $\square$

**Theorem 2.5.** *The spaces  $\int bv$  and  $d(bv)$  are norm isomorphic to  $\ell_1$ .*

*Proof.* We must show that a linear bijection between the spaces  $\int bv$  and  $\ell_1$  exists. Consider the transformation  $T$  defined, with the notation (2.3), from  $\int bv$  to  $\ell_1$  by  $x \mapsto y = Tx$ . The linearity of  $T$  is clear. Also, it is trivial that  $x = \theta$  whenever  $Tx = \theta$  and therefore,  $T$  is injective.

Let  $y \in \ell_1$  and define the sequence  $x = (x_k)$  by  $x_k = \frac{1}{k} \cdot \sum_{j=1}^k y_j$ . Then

$$\|x\|_{\int bv} = \sum_k |\Delta(kx_k)| = \sum_k \left| k \cdot \frac{1}{k} \sum_{j=1}^k y_j - (k-1) \cdot \frac{1}{k-1} \sum_{j=1}^{k-1} y_j \right| = \sum_k |y_k| = \|y\|_{\ell_1} < \infty.$$

Then, we have that  $x \in \int bv$ . So,  $T$  is surjective and norm preserving. Hence  $T$  is a linear bijection. It shown us that the space  $\int bv$  is norm isomorphic to  $\ell_1$ .

As similar, using the notation (2.4), we can define the transformation  $S$  from  $d(bv)$  and  $\ell_1$  by  $x \mapsto y = Sx$ . And also, if we choose the sequence  $x = (x_k)$  by  $x_k = k \cdot \sum_{j=2}^k y_j$  while  $y \in \ell_1$ , then we obtain the space  $d(bv)$  is norm isomorphic to  $\ell_1$  with the norm  $\|x\|_{d(bv)}$ .  $\square$

**Theorem 2.6.** *The spaces  $\int bv$  and  $d(bv)$  have monotone norm.*

*Proof.* Let  $x = (x_k) \in \int bv$ . We define the norms  $\|x\|_{\int bv} = \sum_k |\Delta(kx_k)|$  and  $\|x^{[n]}\|_{\int bv} = \sum_{k=1}^n |\Delta(kx_k)|$ , for all  $x \in \int bv$ . For  $n < m$ ,

$$\|x^{[n]}\| = \sum_{k=1}^n |\Delta(kx_k)| \leq \sum_{k=1}^m |\Delta(kx_k)| = \|x^{[m]}\|,$$

that is,

$$(2.5) \quad \|x^{[m]}\| \geq \|x^{[n]}\|.$$

The sequence  $\|x^{[n]}\|$  is monotonically increasing sequence and bounded above.

$$(2.6) \quad \sup \|x^{[n]}\| = \sup \left( \sum_{k=1}^n |\Delta(kx_k)| \right) = \left( \sum_{k=1}^n |\Delta(kx_k)| \right) = \|x\|.$$

From (2.5) and (2.6), it follows that the space  $\int bv$  has the monotone norm.

In similar way, we can obtain to the space  $d(bv)$  has the monotone norm.  $\square$

Because of the isomorphisms  $T$  and  $S$ , defined in the proof of Theorem 2.5, are onto the inverse image of the basis  $\{e^{(k)}\}_{k \in \mathbb{N}}$  of the space  $\ell_1$  is the basis of the spaces  $\int bv$  and  $d(bv)$ . Therefore, we have the following:

**Theorem 2.7.** (i). Define a sequence  $t^{(k)} = \{t_n^{(k)}\}_{n \in \mathbb{N}}$  of elements of the space  $\int bv$  for every fixed  $k \in \mathbb{N}$  by

$$t_n^{(k)} = \begin{cases} 1/k & , \quad (n \geq k) \\ 0 & , \quad (n < k) \end{cases}$$

Therefore, the sequence  $\{t^{(k)}\}_{k \in \mathbb{N}}$  is a basis for the space  $\int bv$  and if we choose  $E_k = (Ax)_k$  for all  $k \in \mathbb{N}$ , where the matrix  $A$  defined by (1.3), then any  $x \in \int bv$  has a unique representation of the form

$$x = \sum_k E_k b^{(k)}.$$

(ii). Define a sequence  $s^{(k)} = \{s_n^{(k)}\}_{n \in \mathbb{N}}$  of elements of the space  $d(bv)$  for every fixed  $k \in \mathbb{N}$  by

$$s_n^{(k)} = \begin{cases} k & , \quad (n \geq k) \\ 0 & , \quad (n < k) \end{cases}$$

Therefore, the sequence  $\{s^{(k)}\}_{k \in \mathbb{N}}$  is a basis for the space  $d(bv)$  and if we choose  $F_k = (Bx)_k$  for all  $k \in \mathbb{N}$ , where the matrix  $B$  defined by (1.4), then any  $x \in d(bv)$  has a unique representation of the form

$$x = \sum_k F_k b^{(k)}.$$

The result follows from fact that if a space has a Schauder basis, then it is separable. Hence, we can give following corollary:

**Corollary 2.8.** The spaces  $\int bv$  and  $d(bv)$  are separable.

### 3. THE $\alpha$ -, $\beta$ - AND $\gamma$ - DUALS OF THE SPACES $\int bv$ AND $d(bv)$

In this section, we state and prove the theorems determining the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of the sequence spaces  $\int bv$  and  $d(bv)$ .

Let  $x$  and  $y$  be sequences,  $X$  and  $Y$  be subsets of  $\omega$  and  $A = (a_{nk})_{n,k=0}^\infty$  be an infinite matrix of complex numbers. We write  $xy = (x_k y_k)_{k=0}^\infty$ ,  $x^{-1} * Y = \{a \in \omega : ax \in Y\}$  and  $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$  for the *multiplier space* of  $X$  and  $Y$ . In the special cases of  $Y = \{\ell_1, cs, bs\}$ , we write  $x^\alpha = x^{-1} * \ell_1$ ,  $x^\beta = x^{-1} * cs$ ,  $x^\gamma = x^{-1} * bs$  and  $X^\alpha = M(X, \ell_1)$ ,  $X^\beta = M(X, cs)$ ,  $X^\gamma = M(X, bs)$  for the  $\alpha$ -dual,  $\beta$ -dual,  $\gamma$ -dual of  $X$ . By  $A_n = (a_{nk})_{k=0}^\infty$  we denote the sequence in the  $n$ -th row of  $A$ , and we write  $A_n(x) = \sum_{k=0}^\infty a_{nk} x_k$   $n = (0, 1, \dots)$  and  $A(x) = (A_n(x))_{n=0}^\infty$ , provided  $A_n \in x^\beta$  for all  $n$ .

**Lemma 3.1.** [1, Theorem 2.1] Let  $\lambda, \mu$  be the BK-spaces and  $B_\mu^U = (b_{nk})$  be defined via the sequence  $\alpha = (\alpha_k) \in \mu$  and triangle matrix  $U = (u_{nk})$  by

$$b_{nk} = \sum_{j=k}^n \alpha_j u_{nj} v_{jk}$$

for all  $k, n \in \mathbb{N}$ . Then, the inclusion  $\mu \lambda_U \subset \lambda_U$  holds if and only if the matrix  $B_\mu^U = U D_\alpha U^{-1}$  is in the classes  $(\lambda : \lambda)$ , where  $D_\alpha$  is the diagonal matrix defined by  $[D_\alpha]_{nn} = \alpha_n$  for all  $n \in \mathbb{N}$ .

**Lemma 3.2.** [1, Theorem 3.1]  $B_\mu^U = (b_{nk})$  be defined via a sequence  $a = (a_k) \in \omega$  and inverse of the triangle matrix  $U = (u_{nk})$  by

$$b_{nk} = \sum_{j=k}^n a_j v_{jk}$$

for all  $k, n \in \mathbb{N}$ . Then,

$$\lambda_U^\beta = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}.$$

and

$$\lambda_U^\gamma = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.$$

**Lemma 3.3.** Let  $A = (a_{nk})$  be an infinite matrix. Then, the following statements hold:

(i)  $A \in (\ell_1 : \ell_\infty)$  if and only if

$$(3.1) \quad \sup_{k, n \in \mathbb{N}} |a_{nk}| < \infty.$$

(ii)  $A \in (\ell_1 : c)$  if and only if (3.1) holds, and there are  $\alpha_k \in \mathbb{C}$  such that

$$(3.2) \quad \lim_{n \rightarrow \infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}.$$

(iii)  $A \in (\ell_1 : \ell_1)$  if and only if

$$(3.3) \quad \sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty.$$

**Theorem 3.4.**  $[\int bv]^\alpha = d(\ell_1)$

*Proof.* We take the matrix  $\Gamma$  as defined by (1.3) and  $\Gamma_n$  denotes the sequences in the  $n$ th rows of the matrices  $\Gamma$ . We define the matrix  $C$  whose rows are the product of the rows of the matrix  $\Gamma^{-1}$  and the sequence  $a = (a_n)$ , i.e.,  $C_n = (\Gamma^{-1})_n a$ . From the relation (2.3), we obtain

$$(3.4) \quad a_n x_n = \sum_{k=1}^n \frac{a_n}{n} y_k = (Cy)_n \quad (n \in \mathbb{N}).$$

It follows from (3.4) that  $ax = (a_n x_n) \in \ell_1$  whenever  $x = (x_k) \in \int bv$  if and only if  $Cy \in \ell_1$  whenever  $y \in \ell_1$ . By using Lemma 3.3 (iii), we obtain that  $[\int bv]^\alpha = d(\ell_1)$ .  $\square$

**Theorem 3.5.**  $[d(bv)]^\alpha = \int \ell_1$

*Proof.* As similar way in proof of Theorem 3.4, if we take the matrix  $\Sigma$  as defined by (1.4) and define the matrix  $D = (d_{nk})$  with  $a_n x_n = \sum_{k=2}^n n \cdot a_n \cdot y_k = (Dy)_n$  for all  $n \in \mathbb{N}$ , using by the relation (2.4), this gives us that  $[d(bv)]^\alpha = \int \ell_1$ .  $\square$

**Theorem 3.6.**  $[\int bv]^\beta = d(bs)$

*Proof.* Consider the equation

$$(3.5) \quad \sum_{k=1}^n a_k x_k = \sum_{k=1}^n a_k \left( k^{-1} \sum_{j=1}^k y_j \right) = \sum_{k=1}^n \left( \sum_{j=k}^n \frac{a_j}{j} \right) y_k = (Ey)_n$$

where  $E = (e_{nk})$  is defined by

$$(3.6) \quad e_{nk} = \begin{cases} \sum_{j=k}^n j^{-1} a_j & , \quad (0 \leq k \leq n) \\ 0 & , \quad (k > n) \end{cases}$$

for all  $n, k \in \mathbb{N}$ . Then we deduce from Lemma 3.3 (ii) with (3.5) that  $ax = (a_k x_k) \in cs$  whenever  $x = (x_k) \in \int bv$  if and only if  $Ey \in c$  whenever  $y = (y_k) \in \ell_1$ . Thus,  $(a_k) \in cs$  and  $(a_k) \in d(bs)$  by (3.1) and (3.2), respectively. Since the inclusion  $d(bs) \subset cs$  holds, then, we have  $(a_k) \in d(bs)$ , whence  $[\int bv]^\beta = d(bs)$ .  $\square$

**Lemma 3.7.**  $[4] \quad (cs)^\beta = bv \Rightarrow [d(cs)]^\beta = \int bv$

From Theorem 3.6 and Lemma 3.7, we have,

**Theorem 3.8.**  $(bv)^\beta = cs \Rightarrow [d(bv)]^\beta = \int cs$ .

**Theorem 3.9.**  $[\int bv]^\gamma = d(bs)$

*Proof.* This can be obtained by analogy with the proof of Theorem 3.6 with Lemma 3.3 (i) instead of Lemma 3.3 (ii). So we omit the details.  $\square$

**Theorem 3.10.**  $[d(bs)]^\gamma = \int bv$

#### 4. MATRIX MAPPINGS ON THE SPACES $\int bv$ AND $d(bv)$

In this section, we characterize some matrix transformations on the spaces  $\int bv$  and  $d(bv)$ .

We shall write throughout for brevity that

$$\begin{aligned} \bar{a}_{nk} &= k^{-1} \sum_{j=k}^{\infty} a_{nj}, & \tilde{a}_{nk} &= k \sum_{j=k}^{\infty} a_{nj}, \\ \hat{a}_{nk} &= n \cdot a_{nk} - (n-1) \cdot a_{n-1,k}, & \vec{a}_{nk} &= n^{-1} \cdot a_{nk} - (n-1)^{-1} \cdot a_{n-1,k} \end{aligned}$$

for all  $k, n \in \mathbb{N}$ .

**Lemma 4.1.** [1] *Let  $X, Y$  be any two sequence spaces,  $A$  be an infinite matrix and  $U$  a triangle matrix matrix. Then,  $A \in (X : Y_U)$  if and only if  $UA \in (X : Y)$ .*

**Theorem 4.2.** Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $F = (f_{nk})$  are connected with the relation

$$(4.1) \quad f_{nk} = \overline{a}_{nk}$$

for all  $k, n \in \mathbb{N}$  and  $Y$  be any given sequence space. Then,  $A \in (\int bv : Y)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in [\int bv]^\beta$  for all  $n \in \mathbb{N}$  and  $F \in (\ell_1 : Y)$ .

*Proof.* Let  $Y$  be any given sequence space. Suppose that (4.1) holds between  $A = (a_{nk})$  and  $F = (f_{nk})$ , and take into account that the spaces  $\int bv$  and  $\ell_1$  are norm isomorphic.

Let  $A \in (\int bv : Y)$  and take any  $y = (y_k) \in \ell_1$ . Then  $\Gamma F$  exists and  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  which yields that (4.1) is necessary and  $\{f_{nk}\}_{k \in \mathbb{N}} \in (\ell_1)^\beta$  for each  $n \in \mathbb{N}$ . Hence,  $Fy$  exists for each  $y \in \ell_1$  and thus

$$\sum_k f_{nk} y_k = \sum_k a_{nk} x_k \quad \text{for all } n \in \mathbb{N},$$

we obtain that  $Fy = Ax$  which leads us to the consequence  $F \in (\ell_1 : Y)$ .

Conversely, let  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for each  $n \in \mathbb{N}$  and  $F \in (\ell_1 : Y)$  hold, and take any  $x = (x_k) \in \int bv$ . Then,  $Ax$  exists. Therefore, we obtain from the equality

$$\sum_{k=1}^m a_{nk} x_k = \sum_{k=1}^m \left[ k^{-1} \sum_{j=k}^m a_{nj} \right] y_k \quad \text{for all } m, n \in \mathbb{N}$$

as  $m \rightarrow \infty$  that  $Ax = Fy$  and this shows that  $F \in (\ell_1 : Y)$ . This completes the proof.  $\square$

**Theorem 4.3.** Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $G = (g_{nk})$  are connected with the relation

$$(4.2) \quad g_{nk} = \widetilde{a}_{nk}$$

for all  $k, n \in \mathbb{N}$  and  $Y$  be any given sequence space. Then,  $A \in (d(bv) : Y)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in [d(bv)]^\beta$  for all  $n \in \mathbb{N}$  and  $G \in (\ell_1 : Y)$ .

**Theorem 4.4.** Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $H = (h_{nk})$  are connected with the relation

$$(4.3) \quad h_{nk} = \widehat{a}_{nk}$$

for all  $k, n \in \mathbb{N}$  and  $Y$  be any given sequence space. Then,  $A \in (Y : \int bv)$  if and only if  $M \in (Y : \ell_1)$ .

*Proof.* Let  $z = (z_k) \in Y$  and consider the following equality

$$\sum_{k=0}^m \widehat{a}_{nk} z_k = \sum_{k=0}^m (n \cdot a_{nk} - (n-1)a_{n-1,k}) z_k \quad \text{for all } m, n \in \mathbb{N}$$

which yields that as  $m \rightarrow \infty$  that  $(Hz)_n = \{\Gamma(Az)\}_n$  for all  $n \in \mathbb{N}$ . Therefore, one can observe from here that  $Az \in \int bv$  whenever  $z \in Y$  if and only if  $Hz \in \ell_1$  whenever  $z \in Y$ .  $\square$

**Theorem 4.5.** Suppose that the entries of the infinite matrices  $A = (a_{nk})$  and  $M = (m_{nk})$  are connected with the relation

$$(4.4) \quad m_{nk} = \vec{a}_{nk}$$

for all  $k, n \in \mathbb{N}$  and  $Y$  be any given sequence space. Then,  $A \in (Y : d(bv))$  if and only if  $F \in (Y : \ell_1)$ .

**Lemma 4.6.** (i)  $A \in (\ell_1 : bs)$  if and only if

$$(4.5) \quad \sup_{k, m \in \mathbb{N}} \left| \sum_{n=0}^m a_{nk} \right| < \infty.$$

(ii)  $A \in (\ell_1 : cs)$  if and only if (4.5) holds, and

$$(4.6) \quad \sum_n a_{nk} \text{ convergent for each } k \in \mathbb{N}.$$

(iii)  $A \in (\ell_1 : c_0s)$  if and only if (4.5) holds, and

$$(4.7) \quad \sum_n a_{nk} = 0 \text{ for each } k \in \mathbb{N}.$$

**Lemma 4.7.** (i)  $A \in (\ell_\infty : \ell_1) = (c : \ell_1) = (c_0 : \ell_1)$  if and only if

$$(4.8) \quad \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n, k+1}) \right| < \infty$$

(ii)  $A \in (bs : \ell_1)$  if and only if

$$(4.9) \quad \lim_k a_{nk} = 0 \text{ for each } n \in \mathbb{N}.$$

$$(4.10) \quad \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n, k+1}) \right| < \infty$$

(iii)  $A \in (cs : \ell_1)$  if and only if

$$(4.11) \quad \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n, k-1}) \right| < \infty$$

(iv)  $A \in (c_0s : \ell_1)$  if and only if (4.10) holds.

Now, we can give the following results:

**Corollary 4.8.** *The following statements hold:*

- (i)  $A = (a_{nk}) \in (\int bv : \ell_\infty)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) holds with  $\bar{a}_{nk}$  instead of  $a_{nk}$ .
- (ii)  $A = (a_{nk}) \in (\int bv : c)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) and (3.2) hold with  $\bar{a}_{nk}$  instead of  $a_{nk}$ .
- (iii)  $A \in (\int bv : c_0)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) and (3.2) hold with  $\alpha_k = 0$  as  $\bar{a}_{nk}$  instead of  $a_{nk}$ .
- (iv)  $A = (a_{nk}) \in (\int bv : bs)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5) holds with  $\bar{a}_{nk}$  instead of  $a_{nk}$ .
- (v)  $A = (a_{nk}) \in (\int bv : cs)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5), (4.6) hold with  $\bar{a}_{nk}$  instead of  $a_{nk}$ .
- (vi)  $A = (a_{nk}) \in (\int bv : c_0s)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5), (4.7) hold with  $\bar{a}_{nk}$  instead of  $a_{nk}$ .

**Corollary 4.9.** *The following statements hold:*

- (i)  $A = (a_{nk}) \in (d(bv) : \ell_\infty)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv)\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) holds with  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .
- (ii)  $A = (a_{nk}) \in (d(bv) : c)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv)\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) and (3.2) hold with  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .
- (iii)  $A \in (d(bv) : c_0)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv)\}^\beta$  for all  $n \in \mathbb{N}$  and (3.1) and (3.2) hold with  $\alpha_k = 0$  as  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .
- (iv)  $A = (a_{nk}) \in (d(bv) : bs)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv)\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5) holds with  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .
- (v)  $A = (a_{nk}) \in (d(bv) : cs)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{d(bv)\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5), (4.6) hold with  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .
- (vi)  $A = (a_{nk}) \in (d(bv) : c_0s)$  if and only if  $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\int bv\}^\beta$  for all  $n \in \mathbb{N}$  and (4.5), (4.7) hold with  $\tilde{a}_{nk}$  instead of  $a_{nk}$ .

**Corollary 4.10.** *We have:*

- (i)  $A = (a_{nk}) \in (\ell_\infty : \int bv) = (c : \int bv) = (c_0 : \int bv)$  if and only if (4.8) hold with  $\hat{a}_{nk}$  instead of  $a_{nk}$ .
- (ii)  $A = (a_{nk}) \in (bs : \int bv)$  if and only if (4.9) and (4.10) hold with  $\hat{a}_{nk}$  instead of  $a_{nk}$ .
- (iii)  $A = (a_{nk}) \in (cs : \int bv)$  if and only if (4.11) holds with  $\hat{a}_{nk}$  instead of  $a_{nk}$ .
- (iv)  $A = (a_{nk}) \in (c_0s : \int bv)$  if and only if (4.10) holds with  $\hat{a}_{nk}$  instead of  $a_{nk}$ .

**Corollary 4.11.** *We have:*

- (i)  $A = (a_{nk}) \in (\ell_\infty : d(bv)) = (c : d(bv)) = (c_0 : d(bv))$  if and only if (4.8) hold with  $\vec{a}_{nk}$  instead of  $a_{nk}$ .
- (ii)  $A = (a_{nk}) \in (bs : d(bv))$  if and only if (4.9) and (4.10) hold with  $\vec{a}_{nk}$  instead of  $a_{nk}$ .
- (iii)  $A = (a_{nk}) \in (cs : d(bv))$  if and only if (4.11) holds with  $\vec{a}_{nk}$  instead of  $a_{nk}$ .
- (iv)  $A = (a_{nk}) \in (c_0s : d(bv))$  if and only if (4.10) holds with  $\hat{a}_{nk}$  instead of  $a_{nk}$ .

## 5. CONCLUSION

Goes and Goes [4] introduced the integrated and differentiated sequence spaces. Subramanian et.al. [5] gave the integrated rate space  $\int \ell_\pi$  and studied some properties of this space. And they also characterized the matrix classes  $(\int \ell_\pi : Y)$ , where  $Y = \{\ell_\infty, c, c_0, \ell_p, bv, bv_0, bs, cs, \ell_\rho, \ell_\pi\}$ . There are no studies on differentiated sequence spaces.

In this paper, we studied some properties of integrated and differentiated sequence spaces. We compute the alpha-, beta- and gamma-duals of these spaces. For  $Y = \{\ell_\infty, c, c_0, bs, cs, c_0s\}$ , we characterize matrix classes  $(\int bv : Y)$ ,  $(d(bv) : Y)$  and  $(Y : \int bv)$ ,  $(Y : d(bv))$  in the last section.

We should note from now on that the investigation of the domain of some particular limitation matrices, namely Cesàro means of order one, Euler means of order  $r$ , Riesz means, Nörlund means, the double band matrix  $B(r, s)$ , the triple band matrix  $B(r, st)$ , etc., in the spaces  $\int bv$  and  $d(bv)$  will lead us to new results which are not comparable with the present results. If we can choose different sequence spaces for the space  $Y$ , it can study new matrix characterizations of  $(\int bv : Y)$ ,  $(d(bv) : Y)$  and  $(Y : \int bv)$ ,  $(Y : d(bv))$ . Also the spaces  $\int bv$  and  $d(bv)$  can be defined by a index  $p$  and paranormed sequence spaces as  $p = (p_k)$  is a sequence of strictly positive numbers.

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